Encouraging Teachers to Make Use of Multiplicative Structure

Ziv Feldman
Boston University, Massachusetts

Matt B. Roscoe
University of Montana

The literature has shown that preservice elementary school teachers (PSTs) struggle to adequately attend to a number's multiplicative structure to determine divisibility. This study describes an intervention aimed at strengthening preservice and in-service teachers' procedural knowledge with respect to using a number's prime factorization to identify its factors, and presents evidence of the impact of the intervention. Results point toward improved abilities to use a number's prime factorization to sort factors and nonfactors across four factor subtypes, to create factor lists, and to construct numbers with particular divisibility properties. Implications for mathematics teacher education include providing specific materials and strategies for strengthening preservice and in-service teachers' procedural knowledge.

Keywords: Procedural knowledge, Teacher preparation, Multiplicative structure, Divisibility, Prime factorization

Introduction

Mathematics teacher educators should ask preservice and in-service elementary school teachers questions that support their ability to make sense of mathematics (Conference Board of the Mathematical Sciences [CBMS], 2012). Such questions should include both how and why mathematical procedures work. With this in mind, consider the following two questions:

Q1: Find all the factors of the number 360.

Q2: Find all the factors of the number $2^3 \times 3^2 \times 5$.

The first question, commonly asked of students in elementary school, is often answered using an exhaustive method where one trial divides 360 by 1, 2, 3, 4, etc., up to the point where the quotient is less than the divisor. Those integers that divide 360 (along with their corresponding quotients) are identified as factors. Those integers that do not divide 360 are identified as nonfactors. The second question, one that is less commonly asked of students in elementary school, can be answered through an application of the fundamental theorem of arithmetic (FTA). This foundational result in number theory states that every positive integer (except for the number 1) can be represented as a product of one or more primes in exactly one way, apart from rearrangement (Hardy & Wright, 1979, pp. 2–3). An application of the FTA to Q2 allows us to discover that, for example, $2^2 \times 3 = 12$ is a factor because it can be multiplied by another integer $(2 \times 3 \times 5) = 30$ to arrive at the given number: $(2^2 \times 3) \times (2 \times 3 \times 5) = 2^3 \times 3^2 \times 5$. Similarly, $(2 \times 3 \times 5)$ must also be a factor of the given number. The FTA can also help us determine that $21 = 7 \times 3$ is not a factor because the given prime factorization, which is the only one that exists for this number, does not include a 7, which is a necessary prime factor of 21. Using the commutative and associative properties of multiplication, one can find every possible subset of the number's prime factors included in its prime factorization. The product of each of these combinations represents a factor of the original number; in this fashion, we arrive at the same answer to Q2 as for Q1.

The point of this exercise is to consider the importance of asking both questions of those who are preparing to teach mathematics. The first question is, no doubt, important—a generalization of the procedure that leads to its answer is a fundamental skill that helps teachers answer the question, "How can I find a number's factors?" The second question is equally important because, in the application of the FTA to the analysis of factors, teachers discover an answer to the question, "Why are only some numbers factors of a given number?" We argue that teachers who study both of these approaches develop a deeper procedural knowledge (Rittle-Johnson, Star, & Durkin, 2012; Star, 2005) of the multiplicative structure of numbers and, thus, are in a better position to enact instruction that promotes students' procedural fluency (National Research Council, 2001).

Why Focus on Multiplicative Structure?

National reports and existing research identify the importance of developing teachers' knowledge of the mathematics they teach to students in K–grade 12 (Association of Mathematics Teacher Educators [AMTE], 2017; CBMS, 2001, 2012; Greenberg & Walsh, 2008; Ma, 1999; National Mathematics Advisory Panel, 2008). If students in K–grade 8 are to develop mathematical proficiency (National Research Council, 2001), then teachers must develop both knowledge of concepts and relationships and a deep and flexible knowledge of mathematical procedures (Ball, Thames, & Phelps, 2008; Star, 2005). *The Standards for Preparing Teachers of Mathematics* (AMTE, 2017) affirms the importance of attending to both conceptual and procedural knowledge, stating that "well-prepared beginning teachers of mathematics understand and solve problems in more than one way, explain the meanings of key concepts, and explain the mathematical rationales underlying key procedures" (p. 8). However, the research literature on mathematical knowledge has historically prioritized conceptual knowledge above procedural knowledge (Star, 2005). Given its importance in developing students' mathematical proficiency, we take up Star's (2005) call for a renewed focus on procedural knowledge in the context of elementary teacher preparation.

Although various definitions exist, we follow the National Research Council (2001), which refers to procedural knowledge as procedural *fluency* and defines it as "knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121). One indicator of deep procedural knowledge is procedural flexibility, which refers to knowing multiple procedures for solving particular problems and being able to make efficient choices around which of those procedures to use in particular situations (Rittle-Johnson et al., 2012; Star, 2005). Research has shown that the procedural knowledge of many preservice elementary teachers (PSTs) is superficial and disconnected (Browning, Edson, Kimani, & Aslan-Tutak, 2014; Kastberg & Morton, 2014; Thanheiser, Philipp, Fasteen, Strand, & Mills, 2013; Thanheiser, Whitacre, & Roy, 2014). In line with Whitacre and Nickerson (2016), we take the perspective that such characterizations of PSTs' knowledge stem from a lack of opportunity to develop deeper understandings during their K–grade 12 mathematics coursework. In fact, studies do show that PSTs can gain richer understandings of mathematical procedures through instruction (Feldman, 2012; McClain, 2003; Simon & Blume, 1994; Whitacre & Nickerson, 2016).

One content area that has been particularly challenging for PSTs but is largely ignored in the current literature is the use of multiplicative structure to determine divisibility. We use Zazkis and Campbell's (1996a) definition of multiplicative structure: the "conceptual attributes and relations pertaining to and implied by the decomposition of natural numbers as unique products of prime factors" (p. 541). Zazkis and Campbell (1996a, 1996b) showed that PSTs exhibit "procedural attachments" when solving divisibility problems. When asked to identify several factors of $M = 3^3 \times 5^2 \times 7$, the majority of PSTs first computed the whole number value of M and then used trial division to test possible factors. Some research suggests that PSTs' procedural attachments are due to an inability or unwillingness to treat a number's prime-factored form as an equivalent quantity (Zazkis & Gadowsky, 2001). However, these supposed deficiencies may be the result of insufficient opportunities for PSTs to work with prime-factored form (Feldman, 2012; Zazkis & Campbell, 1996b). Even instruction that successfully emphasizes rote procedure may impede PSTs' use of multiplicative structure, thereby inhibiting stronger schemas of multiplication and division concepts (Brown, Thomas, & Tolias, 2002).

This research points to a pressing need for mathematics teacher educators to support PSTs' attention to multiplicative structure, including its core underlying concept, the fundamental theorem of arithmetic (FTA). First, multiplicative structure provides opportunities to examine several foundational mathematical ideas and properties that elementary school teachers need to know and be able to teach, such as prime and composite numbers and the commutative and associative properties (Griffiths, 2013). These foundational topics form the basis for students' understanding and use of a wide range of arithmetic operations and computational strategies (National Council of Teachers of Mathematics [NCTM], 2000; National Research Council, 2001). In fact, Griffiths (2013) refers to the FTA as "one of the cornerstones of elementary mathematics in that it underpins anything associated with the multiplicative properties of the integers . . ." (p. 78). Second, multiplicative structure is embedded in more advanced topics on the educational horizon, such as greatest common factor and least common multiple, rational number operations, polynomial functions, and rational functions (Feldman, 2014; NCTM, 2000; Vergnaud, 1988).

Attending to multiplicative structure also provides much needed opportunities for PSTs to make use of structure, a mathematical practice described in the Common Core State Standards (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). A focus on structure includes looking for patterns and decomposing systems into their component parts. Decomposing numbers multiplicatively brings our number systems' building blocks, prime numbers, to the foreground and provides rich opportunities for conjecture-making and generalizing (Cuoco, Goldenberg, & Mark, 1996). Examining mathematical structure also supports richer understandings, connects concepts to procedures, and shifts student learning away from memorizing (Mason, Stephens, & Watson, 2009). Structure can also help teachers recognize the limitations of overemphasizing rote procedures during instruction (Vale, McAndrew, & Krishnan, 2011). As such, there are likely both mathematical and pedagogical affordances in addressing multiplicative structure in content courses for PSTs.

In this article, we describe the impact of an intervention aimed at deepening preservice and in-service elementary teachers' procedural knowledge of the multiplicative structure of number as it pertains to divisibility. The intervention makes central the role of prime factorization as a tool for promoting such understandings.

Background Literature

This study adopts a constructivist perspective on learning (Dubinsky, 1991; Piaget, 1985; von Glasersfeld, 1987). Learners construct new knowledge by organizing their experiences through processes of assimilation and accommodation (Piaget, 1985). The learner may assimilate a new experience into prior knowledge, or if a new experience conflicts with what he or she already knows, the learner may accommodate that new experience by revising previous understandings.

Following this perspective, prior research has focused on PSTs' difficulties in using prime factorization to solve divisibility problems. Most of the research in this area has been conducted by Rina Zazkis and her colleagues. For example, interviews with 21 PSTs found that, when asked to find the factors of a number expressed in prime-factored form, the majority of participants first computed the number's whole number value and then performed trial division (Zazkis & Campbell, 1996a). Zazkis and Gadowsky (2001) framed this finding as an inability to use the transparent features of prime factorization (e.g., the prime-factored representation of $N = 2^3 \times 3^2 \times 5^3$ makes it transparent that 5 is a factor of N and that 7 is not a factor of N). Brown et al. (2002) called for pedagogical interventions to emphasize PSTs' flexible reasoning with numbers written in prime-factored form.

Zazkis and colleagues also sought to characterize the nature of PSTs' knowledge of number theory topics. When confronted with a number written in prime-factored form, PSTs tend to more easily identify a number's factors than its nonfactors and a number's prime factors than its composite factors (Zazkis & Campbell, 1996a, 1996b). Zazkis and Campbell (1996b) identified a lack of appreciation for the uniqueness feature of the FTA as a possible explanation: "Whereas the existence of prime decomposition may be taken for granted, the uniqueness of prime decomposition appears to be counterintuitive and often a possibility of different prime decompositions is assumed" (p. 217).

Only a handful of studies have examined the efficacy of classroom interventions aimed at improving PSTs' knowledge of divisibility and prime factorization. Feldman (2012) implemented a set of number theory tasks focused on the use of prime factorization with 59 preservice elementary teachers. He found that their ability to identify factors and nonfactors, as well as solve greatest common factor (GCF) and least common multiple (LCM) problems, improved significantly. Sinclair, Zazkis, and Liljedahl (2004) and Liljedahl, Sinclair, and Zazkis (2006) used a computer applet in which preservice elementary teachers used an interactive array of whole numbers to explore visual patterns generated by factors and multiples. The researchers found that the combined effects of visualization and experimentation led to a more robust understanding of the multiplicative structure of natural numbers, primes, composites, and evens and odds.

Despite the focused research on PSTs' knowledge of prime factorization and divisibility concepts, little is known about the kinds of mathematical tasks that can best support their learning. Incorporating the successes described in previous studies, this paper presents a sequence of three instructional lessons that is meant to promote teachers' ability to use prime factorization to solve divisibility problems.

The Intervention

Our intervention is a sequence of three instructional lessons and two accompanying homework assignments loosely adapted from an NSF-funded project[1] in which the first author was a co-principal investigator. The overarching aim of the intervention is to strengthen participants' ability to recognize and make use of a number's multiplicative structure, which is transparently provided by its prime factorization. Each lesson consists of cycles of problem sets and discussion questions. Each cycle consists of short sets of interrelated mathematical problems designed to be cognitively demanding: Participants are asked to connect procedures with underlying concepts, make and test their own conjectures, and provide reasoning for their ideas (Stein, Smith, Henningsen, & Silver, 2009). Participants work in small groups of three

1 Original lesson materials were developed by a team of mathematics teacher educators at Boston University as part of the *Elementary Preservice Teachers' Mathematics Project: Developing Faculty Expertise*, under NSF grant no. TUES-1323156.

or four to complete one set of problems, as the instructor (each author) circulates from group to group observing, listening, and interrupting only to ask guiding and probing questions. Each set is followed by whole-class discussion questions meant to help groups synthesize and articulate their thinking about the preceding set of problems. During discussions, the instructor encourages participants to explain and justify their own mathematical thinking and avoids providing solutions for them. Following discussion, a new cycle consisting of another short set of problems and discussion questions begins. All three lessons and their associated homework assignments are provided in **Appendix A**.

The first lesson is a 30-minute lesson introducing participants to the fundamental theorem of arithmetic by exploring factor trees and factored forms of whole numbers. To make sense of the FTA, participants are first asked to construct different factor trees for the same number (see Figure 1) and then explain how these factor trees are similar to and different from one another. They are then asked to compare three numbers written as (nonprime) factorizations and to determine, without multiplying, which number has the greatest value.

The purpose of this task is to make salient the uniqueness feature of the FTA. The lesson concludes with two discussion questions. The first question asks participants to discuss the differences between a factorization of a number and a prime factorization of a number. The second question presents a formal definition of the FTA and asks participants to explain the meaning of the theorem in their own words. Here, our goal is to encourage participants to deepen their understanding of the theorem by translating its meaning into a more personal statement.

When enacting this lesson, we find it useful to first introduce the concept of factor trees and then let participants explore the remaining problems in groups with minimal instructor interference. Although most participants do not struggle to explain the meaning of the FTA, we recognize that simply explaining the theorem does not guarantee understanding (Zazkis & Campbell, 1996b). Given the opportunity to explore and with explicit prompting from the instructor, participants are able to construct useful metaphors that reflect the foundational importance of the FTA (e.g., number DNA or fingerprint).

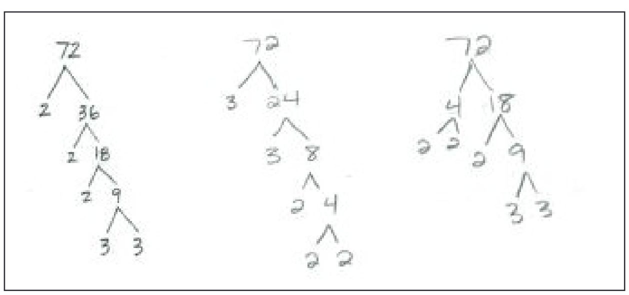


Figure 1. **Using factor trees to find the prime factorization of 72.**

The first lesson is followed by a homework assignment. Participants are given a 10-by-10 gridded array numbered from 1–100 (see Figure 2) and asked to fill each grid square with the corresponding prime factorization. Participants identify patterns in the array and any shortcuts to constructing each prime factorization. For example, in Figure 2, one participant discovers that most primes end in either 1, 3, 7, or 9, calling these columns in the grid "prime rows." The array serves as a participant-generated visualization tool that will be employed in Lesson 2 to explore the transparent features of prime-factored form. This approach not only served as useful practice for finding prime factorizations, it also capitalized on the results of previous studies (i.e., Liljedahl et al., 2006; Sinclair et al., 2004) that employed visual array representations to support PSTs' attention to the multiplicative structure of counting numbers.

The goal of Lesson 2 (approximately 90 minutes) is to generate a method for finding the factors of a number using the number's prime factorization. Initially, small groups are assigned a particular number (e.g., 90) and

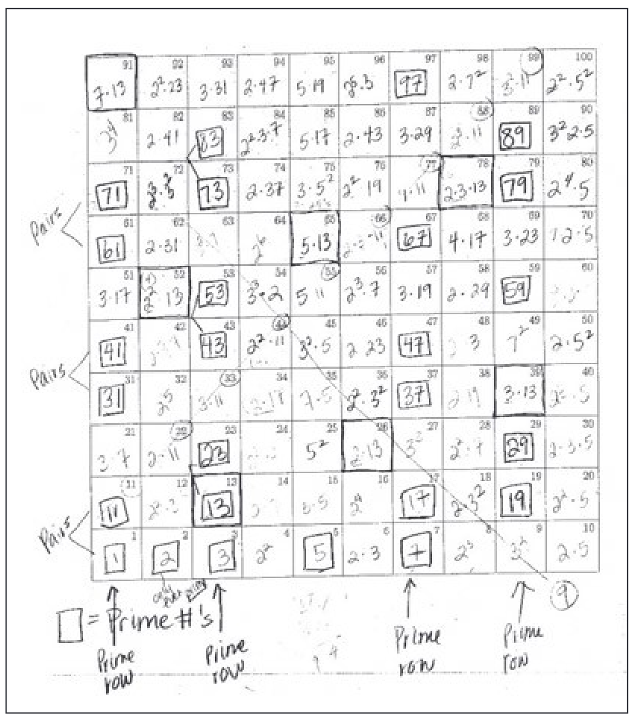
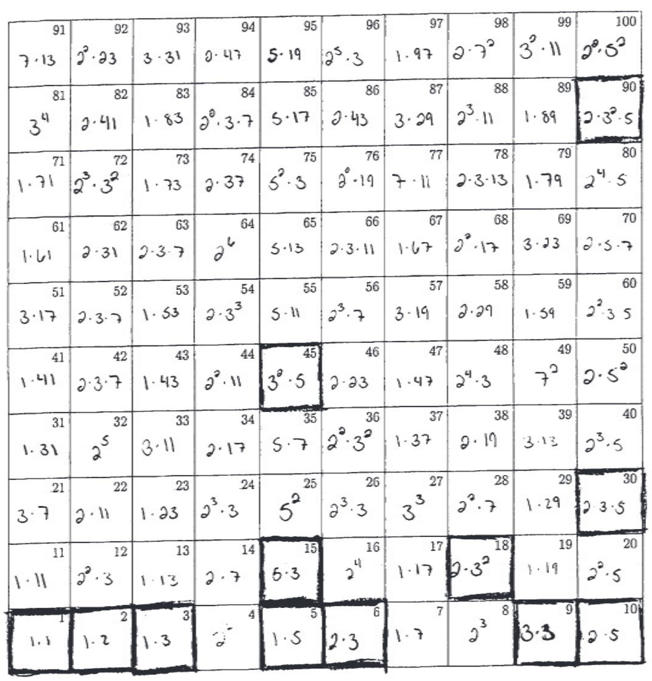


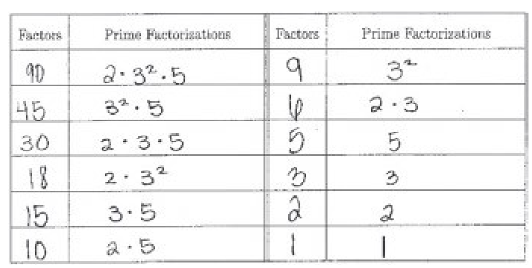
Figure 2. **A completed array showing the prime factorizations of the first 100 counting numbers.**

instructed to use their completed arrays to find its factors in both whole number and prime-factored forms (see Figure 3). Assigning a different number to each group gives participants an opportunity to address a variety of multiplicative structures in subsequent whole-class discussions. We used the following numbers: 90, 96, 84, 80, 78, and 72. Participants use their arrays to make and test conjectures for how the prime factorizations of the factors of a number are related to its prime factorization. The instructor then facilitates a whole-class discussion in which each group presents its conjectures. In our experience, participants can identify the correct relationship but struggle to justify it. Instructors can support participants' reasoning by pressing them to generalize their thinking. Questions such as "I see your conjecture works for 90, but how do you know it works for all numbers?" can help them begin to consider the structural reasons—commutative and associative properties—for why the prime factors of a factor (e.g., $2 \times 3 \times 5$) are present within the prime factorization of its multiple (e.g., $2 \times 3^2 \times 5$). Participants may need to generate many numerical examples before they can construct robust justifications.

Participants then use their conjectures to solve a variety of divisibility problems, including the following: "Let $K = 31^2 \times 83^2$ be the prime factorization of K. List every counting number that is a factor of K without computing the value of K. Explain briefly how to do this." The purpose of this problem is to force participants to attend to the multiplicative structure of a number by prohibiting the use of long division. Participants must build on their work in earlier parts of the lesson to find products of combinations of prime factors (i.e., 31, 83, 31×83, 31^2, 83^2, $31^2 \times 83$, 31×83^2, 1, and $31^2 \times 83^2$). Finding a systematic way to identify every combination is challenging and may result in participants inadvertently skipping over some factors. Nevertheless, we found it useful to wait until the whole-class discussion to address this potential challenge. Questions such as "How do you know you found every factor?" and "How can we keep track of all combinations so we don't miss any?" prompted participants to develop more robust factoring strategies (e.g., identifying factors whose prime factorizations possess exactly 1 prime, 2 primes, 3 primes, and so on).

The two discussion questions at the conclusion of Lesson 2 ask participants to (a) describe a method for finding the factors of a number using the number's prime factorization, and (b) use prime factorization to describe a new way to define a factor of a number. The first question challenges participants' inclination to revert to whole-number long division (i.e., Zazkis, 1998; Zazkis & Campbell, 1996a). The second question attempts to shift participants' notion of factor from one that is linked to





Factors	Prime Factorizations	Factors	Prime Factorizations
90	$2 \cdot 3^2 \cdot 5$	9	3^2
45	$3^2 \cdot 5$	6	$2 \cdot 3$
30	$2 \cdot 3 \cdot 5$	5	5
18	$2 \cdot 3^2$	3	3
15	$3 \cdot 5$	2	2
10	$2 \cdot 5$	1	1

Figure 3. **Example of participant's use of an array to find the factors of 90.**

the division operation to one that attends to a number's multiplicative structure. Throughout these discussions, we found it useful to repeatedly push participants to explain how the FTA supports their thinking when identifying factors and nonfactors. Figure 4 displays responses to both questions from two different participants.

In Lesson 3 (approximately 60 minutes), participants use their knowledge of the relation between a number's prime factorization and its factors to develop a general rule for finding the number of factors of a number using its prime factorization. Participants first generate counting numbers with 2, 3, 4, 5, and 6 factors (see Feldman, 2014; Teppo, 2002) and describe their structures. For example, a number with exactly three distinct factors must be the square of a prime; a number with four factors must be either the cube of a prime or the product of two distinct primes. When enacting this task, we provided participants with a lot of time to generate different numbers and test conjectures. We circulated from group to group and listened for different conjectures. What we heard helped us determine which groups to call on to present during whole-class discussion.

Participants then use this work to find the number of factors of prime powers (e.g., 2^n) and products of two distinct prime powers (e.g., $2^3 \times 3^2$). They recognize that 2^n has $n + 1$ factors and that $2^3 \times 3^2$ has 4×3, or 12, factors. The lesson concludes with participants explaining how it is possible to determine the number of factors of a number from its prime factorization without finding its factors. During this discussion, participants use their prior work with prime powers to make two generalizations: (a) Any prime power, p^n, has $n + 1$ factors since its factors are $p^0, p^1, p^2, p^3, \ldots, p^n$; and (b) any number written as a sequence of prime powers, $p_1^{n_1}, p_2^{n_2}, p_3^{n_3}, p_m^{n_m}$, has $(n_1 + 1)(n_2 + 1)(n_3 + 1) \ldots (n_m + 1)$ factors. During our enactments, we pushed participants to explain why this formula makes sense, with specific attention to why 1 is added to each exponent and why these sums are multiplied together. We have found that some participants reverted back to the systematic approach devised in Lesson 2 for listing out every factor as a way to support this reasoning. Presenting a simpler case (e.g., $2^3 \times 3^2$) and asking groups to devise a visual (e.g., tree diagrams, tables) to show its factors can help facilitate this work.

Homework 2 is the final assignment of the intervention and serves to reinforce participants' ability to attend to multiplicative structure. Each of the seven questions requires participants to show their work and justify their mathematical thinking. An example follows:

A number, X, has 10 and 3 as two of its factors. What other number(s) must be factors of X? How do you know?

Other questions address common student misconceptions about factors that can be resolved by appealing to the multiplicative structure of counting numbers (Zazkis, 1999). An example of this type of question follows:

a) Describe a method for finding the factors of a number using the number's prime factorization	b) A traditional definition of a *factor* of a number *N* is as follows: "The number, *a*, is a factor of *N* if $N = a \cdot b$, where *a* and *b* are whole numbers." Use prime factorization to describe a new way to define a *factor* of a number.
$18 = 2 \cdot 3^2$ Factors 2 3 3^2 1 $2 \cdot 3$ $2 \cdot 3^2$	Every factor of a number N is some subset of N's prime factorization.
$18 = 2 \times 3^2$ $2 \times 3 \times 3$ $(2 \times 3) \times 3$ $(3 \times 3) \times 2$ $(2 \times 3 \times 3) \times 1$ 2, 3, 6, 9, 18	A factor of N is any of the prime factors of N or the product of any of these prime factors.

Figure 4. **Lesson 2 group discussion questions and student responses.**

If a fifth grader states, "Larger numbers have more factors than smaller numbers," what numbers will you give him to investigate? Why those numbers?

Methodology

To investigate the impact of the three-lesson sequence on teachers' use of multiplicative structure, a classroom intervention was conducted at two large universities in two different states. Participants (n = 69) were undergraduate and graduate students enrolled in mathematics content courses for both preservice and in-service elementary and special education teachers. All participants were seeking their teaching license as part of their university coursework. Of the 69 participants, 47 were undergraduate preservice elementary teachers enrolled at one of the two universities in the study. The remaining 22 participants were graduate in-service teachers majoring in either elementary or special education at the second university. Since the majority (68%) of graduate students in the sample completed their undergraduate coursework the prior year and nearly all were in their first year of classroom teaching, we did not distinguish between undergraduate and graduate participants for analysis purposes. Additionally, we did not distinguish between elementary and special education majors because both share the same mathematics content requirements for licensure in the states where the experiment was conducted. We recognize, however, that the grouping of elementary and special education preservice and in-service teachers in this sample may not reflect more general contexts across states.

Participants were asked to complete a pretest before the start of the intervention and a posttest approximately two weeks after the conclusion of the intervention. The intervention consisted of the three in-class lessons and two out-of-class homework assignments described above. The pretest and posttest were developed and scored by the authors for the purposes of answering the research questions. Three identical question types were used on both the pretest and the posttest. Questions were taken or adapted from prior research on teachers' knowledge of multiplicative structure (Zazkis & Campbell, 1996a). Numerical values were changed from pretest to posttest in Questions 1 and 2, but across all questions, participants were asked to show their work or provide their reasoning. Table 1 lists each test question for both the pretest and the posttest. The following research question with three subquestions guided the study:

1. What is the impact of an intervention focusing on the multiplicative structure of number on preservice and in-service elementary teachers' procedural knowledge as it relates to factors and prime factorization?

   (a) Is participation in the intervention associated with improvements in preservice and in-service elementary teachers' abilities to use a number's prime factorization to identify (a) prime factors, (b) prime nonfactors, (c) composite factors, and (d) composite nonfactors?

   (b) Is participation in the intervention associated with improvements in preservice and in-service

Table 1
Pretest and Posttest Questions

No.	Pretest	Posttest
1	Consider the number $N = 3^2 \times 5^4 \times 11 \times 17^3$. **Without calculating the value of *N***, determine whether each of the following is a factor of *N*. Justify each briefly. (a) 5 (b) 19 (c) 15 (d) 21 (e) 75	Consider the number $N = 2^3 \times 5^4 \times 7^2 \times 13$. **Without calculating the value of *N***, determine whether each of the following is a factor of *N*. Justify each briefly. (a) 11 (b) 7 (c) 14 (d) 21 (e) 98
2a	List all the factors of 225. Show how you found all of them.	List all the factors of 300. Show how you found all of them.
2b	List all of the factors of $5^2 \times 7^2$. Show how you found all of them.	List all of the factors of $5^2 \times 7 \times 13^2$. Show how you found all of them.
3	What is the smallest positive integer that has the first ten counting numbers, 1–10, as its factors? Show or explain your work so that others can follow your logic. *Note:* You may leave your answer in factored form.	What is the smallest positive integer that has the first ten counting numbers, 1–10, as its factors? Show or explain your work so that others can follow your logic. *Note:* You may leave your answer in factored form.

elementary teachers' abilities to make use of prime factorization to generate a list of factors?

(c) Is participation in the intervention associated with improvement in preservice and in-service elementary teachers' abilities to use prime factorization to identify numbers with given factors?

Question 1a assesses participants' abilities to determine whether a given number is a factor of a number expressed in prime-factored form. Directions prohibit converting the number's prime factorization to whole number form because the purpose is to assess participants' ability to use a number's prime factorization. The purpose behind Question 1b is to determine the extent to which participants can use prime-factored and whole number forms to determine factors. Within each test, both numbers are intentionally chosen to have the same multiplicative structure to minimize any differences in complexity (e.g., 225 and 52 × 72 are both the product of the squares of two distinct primes). Question 1c is meant to challenge participants to use multiplicative structure to construct a number, given a set of known factors. Constructing a number from known factors is an example of what Dubinsky (1991) refers to as "reversal," a mental construction in which the individual reverses a process to construct a new process. Reversal is challenging because it requires the individual to have already learned the process that needs reversing (Dubinsky, 1991). Adding to the challenge of Question 1c is that the first 10 counting numbers are not relatively prime, so simply multiplying them together does not result in the correct answer.

Test scoring was conducted using a researcher-developed rubric (see **Appendix B**). Each test was scored out of 25 points, of which 14 points (56%) were given for correct numerical answers, and 11 points (44%) were given for clear and accurate reasoning. Both authors determined inter-rater reliability by independently scoring the same subset of the data (21.7% of the data, or 15 of 69 pretests and posttests). Analysis of the independent scoring revealed 82.5% agreement. Discrepancies in scoring were resolved via discussion and rubric clarification until 100% agreement was achieved. Once reliability had been established, the remaining data were divided equally between the two authors and scored separately.

Findings

In this section, we present the results of the study as they pertain to the research question and its three subquestions.

Research Question 1: What is the impact of an intervention focusing on the multiplicative structure of number on preservice and in-service elementary teachers' procedural knowledge as it relates to factors and prime factorization?

Analysis of the data across all pretest and posttest questions revealed that participation in the intervention is associated with an increase in participants' procedural knowledge related to using a number's multiplicative structure to solve problems. After we verified that foundational assumptions were met, a paired sample *t*-test was conducted to compare participants' mean scores on the pretest to their mean scores on the posttest. Results of the *t*-test indicated a significant difference between participants' pretest ($M = 8.81$ [35.2%], $SD = 4.40$) and posttest scores ($M = 17.78$ (71.1%), $SD = 4.97$); $t(68) = -13.88$, $p < 0.05$). Effect size ($d = 1.9$) was also computed for this analysis. The results suggest that participants' mean scores on the posttest were significantly greater than their mean scores on the pretest, and that this difference is large. As such, the intervention appears to have supported participants' abilities to use prime factorization to successfully solve problems related to divisibility. In the following, we share evidence that shows how the participants' responses to the exam questions demonstrate a deeper understanding of the procedures for using prime factorization to identify factors and determine numbers with given factors.

Research Question 1a: Is participation in the intervention associated with improvements in preservice and in-service elementary teachers' abilities to use a number's prime factorization to identify (a) prime factors, (b) prime nonfactors, (c) composite factors, and (d) composite nonfactors?

The data provided more specific information regarding participants' use of multiplicative structure to identify a number's prime factors, prime nonfactors, composite factors, and composite nonfactors. Prior research has shown that PSTs typically find it more challenging to identify nonfactors than factors and composite factors than prime factors (Zazkis & Campbell, 1996a, 1996b). Table 2 shows mean scores, as a percent of available points on the scoring rubric, for Questions 1a–1e on both pretest and posttests. The table shows that participants improved in their ability to identify factors across all divisor types.

Table 2
Mean Scores for Questions 1a–1e

Question	Divisor Type Given	Pretest (%)	Posttest (%)
1a	Prime factor	62.3	83.3
1b	Prime nonfactor	46.4	81.2
1c	Composite factor of form $p_1 \times p_2$	46.4	78.3
1d	Composite nonfactor of form $p_1 \times p_2$	31.9	70.3
1e	Composite factor of form $p_1^2 \times p^2$	38.4	75.4

Differences in participants' success rates on various types of problems diminished following the intervention. Prior to the intervention, participants showed a marked difference in their ability to identify prime (62.3%) versus composite (46.4%) factors. Following the intervention, success rates in identifying prime (83.3%) and composite (78.3%) factors increased although the difference between these two divisor types diminished. The same result occurred even when participants were faced with a more challenging composite factor as in Question 1e. Participants' abilities to identify factors versus nonfactors showed similar trends. Before the intervention, participants were much more proficient at identifying prime factors (62.3%) than prime nonfactors (46.4%), but they were nearly equally proficient in these abilities after the intervention (83.3% and 81.2%, respectively). Similarly, before the intervention, participants were more proficient at identifying composite factors (46.4%) than composite nonfactors (31.9%). After the intervention, participant scores increased for both types, but the differences between the success rates decreased (78.3% for composite factors and 70.3% for composite nonfactors).

Research Question 1b: Is participation in the intervention associated with improvements in preservice and in-service elementary teachers' abilities to make use of prime factorization to generate a list of factors?

Prior research shows that PSTs struggle to use a number's prime factorization to generate its factors and instead convert to whole number form and use long division (Zazkis & Campbell, 1996a). To determine the extent to which the intervention can support preservice and in-service teachers' use of prime-factored form to generate factors, the study analyzed results from Questions 2a and 2b. In these questions, participants were asked to construct all of the factors of a particular number written in whole number (2a) and prime-factored (2b) forms (see Table 1). Participants were awarded credit in Question 2a for finding all possible factors using any method. Participants were awarded credit in Question 2b only if they made use of prime factorization in the construction of their response. Table 3 shows the mean scores, as a percent of available points on the scoring rubric, for both questions.

The pretest results support findings from prior research. Participants had more difficulty identifying factors of numbers in prime-factored form (25.5%) than in whole-number form (40.9%). Following the intervention, participants improved their ability to find the factors of numbers written in prime-factored (69.6%) and whole-number forms (72.5%).

Of greater interest is that the difference in success rates between the two categories diminished to nearly zero from pretest to posttest. Following the intervention, participants were nearly equally adept at finding the factors of a number using either form. In fact, many participants voluntarily used prime-factored form even when it was not explicitly called for, as illustrated by the following participant's response to question 2(a) on the posttest (see Figure 5).

This finding suggests that, with respect to determining divisibility, the intervention may have elevated participants' abilities to use prime-factored form to a level comparable to using whole-number form. Although more research is needed to examine whether an emphasis on multiplicative structure can overcome teachers'

Table 3
Mean Scores for Question 2

Question	Pretest (%)	Posttest (%)
2a	40.9	72.5
2b	25.5	69.6

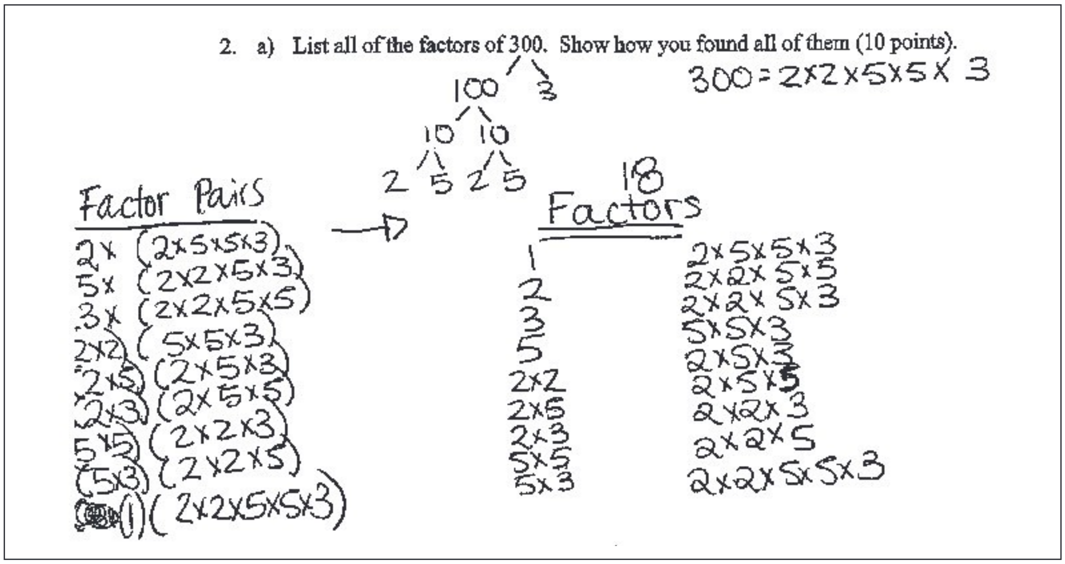


Figure 5. **Using prime factorization to find all of the factors of 300.**

preference for whole-number forms, this finding does suggest that the intervention may support preservice and in-service teachers in leveraging an alternative representation to whole-number form when working with divisibility.

Research Question 1c: Is participation in the intervention associated with improvement in preservice and in-service elementary teachers' abilities to use prime factorization to identify numbers with given factors?

Our analyses thus far have addressed participants' ability to use a number's multiplicative structure to identify and construct factors and nonfactors. Question 3 on the pretest and posttest examines the reverse application of this knowledge—identifying the least number that is divisible by a set of given factors. The question states, "What is the smallest positive integer that has the first ten counting numbers, 1–10, as its factors? Show or explain your work so that others can follow your logic."

Data analysis of Question 3 revealed that participants' ability to construct a number given its factors was extremely weak prior to instruction (19.7% average score). Seventy percent of the participants either made no attempt or erroneously multiplied the first 10 counting numbers together. Approximately 16% of participants were able to identify the correct answer (2,520) but did not provide any reasoning. No participant found the correct answer and provided complete reasoning.

Following instruction, participants' ability to construct a number from given factors improved (58.3% average score). Participants' work was now characterized by the use of prime factorization to determine the answer as well as greater attention to the impact of shared factors on the prime factorization of the answer. Slightly more than 78% found the correct answer; nearly 50% were able to provide complete or nearly complete reasoning. Moreover, participants' reasoning revealed their growing recognition of the relationship between factors and prime factorization. Figure 6 illustrates two participants' responses to Question 3, wherein both explain that the prime factorization of a number must include the prime factorization of each of its factors. Although more research is needed to determine the extent to which participants were able to reconceptualize their prior notions of factor, the data do suggest that attention to multiplicative structure contributed to participants' ability to effectively reverse the process of identifying a number's factors.

Discussion

This study serves as a much-needed example of preservice and in-service elementary and special education teachers' productive mathematical learning. Contrary to prior research (Zazkis & Campbell, 1996a, 1996b; Zazkis & Gadowsky, 2001), participants in this study were capable of attending to multiplicative structure in ways that strengthened their procedural knowledge. Participants were able to coordinate a number's prime factors, identify factors and nonfactors, and reverse that process to construct numbers with known factors. By the end of the intervention, participants also began to reconsider their definition of factor, from "any number that evenly divides" to "any subset of a number's prime factorization." Although prior research often focuses on

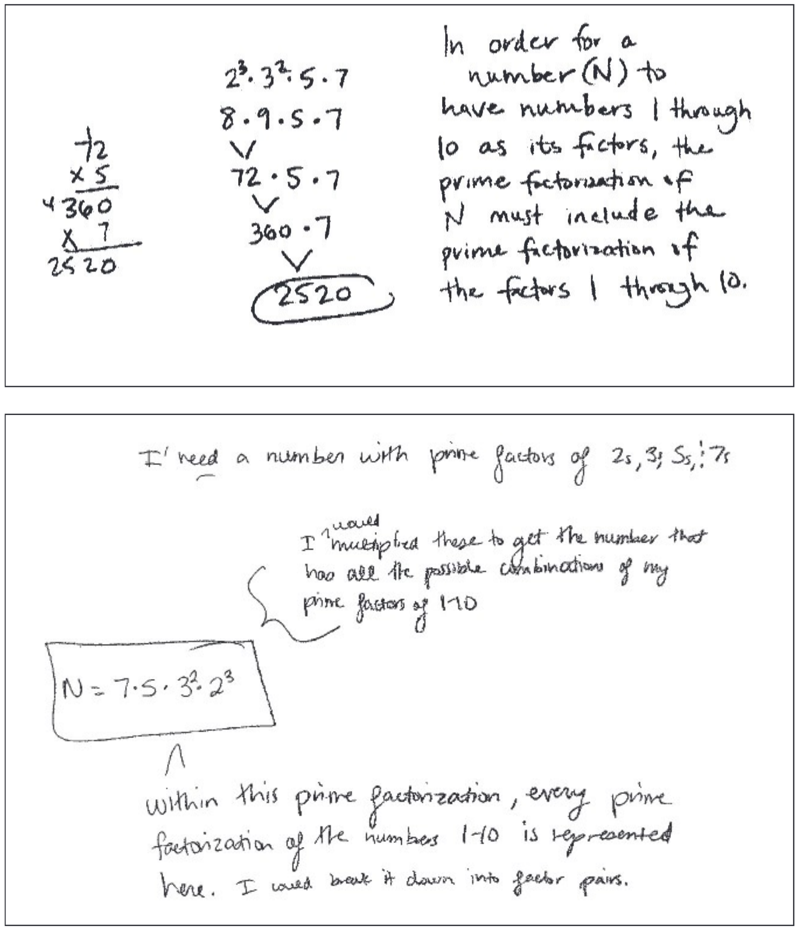


Figure 6. **Articulating the relationship between factors and prime factorization.**

the misconceptions PSTs hold with respect to particular mathematics content (see Thanheiser & Browning, 2014), this study serves as a positive example of the ways in which preservice and in-service teachers' knowledge can grow. Such examples are critical if the field wishes to prepare knowledgeable teachers.

This study also brings to the foreground the potential benefits of encouraging preservice and in-service teachers to grapple with particular numerical representations. Although attending to multiple representations is considered a high-leverage practice (NCTM, 2014), strategically restricting learners' focus to a single representation is not as frequently addressed. We posit that, by restricting their work to prime-factored form, the intervention created a level of uncertainty (Zaslavsky, 2005) for participants that encouraged them to reconsider and, in some cases, refine their previous notions of divisibility.

Participants' consistent use of multiplicative structure and their ability to articulate the reasoning behind its use are evidence of strengthening procedural knowledge of finding a number's factors. By pressing participants to show and explain their thinking using a novel mathematical representation, mathematics teacher educators may provide teachers with greater flexibility in noticing and responding to their own students' thinking. More research is needed, however, to determine whether participants developed the procedural flexibility (Rittle-Johnson et al., 2012) to determine when using one numerical form is more efficient than using the other. When enacting instruction, mathematics teacher educators should consider their lessons' learning goals and their preservice and in-service teachers' prior knowledge to determine whether, and to what extent, a particular representation should be emphasized.

The intervention also provides teacher educators with research-based curriculum materials designed to strengthen preservice and in-service teachers' procedural knowledge of multiplicative structure. The lesson structure—cycles of short subsets of cognitively demanding problems following by discussion questions—creates multiple opportunities to revisit and refine one's mathematical thinking in both small-group and whole-class formats. Because the ways in which lessons are implemented in the classroom impact their quality (Stein et al., 2009), teacher educators looking to use these materials should give practicing and future teachers a significant amount of time to explore problems in small groups and share their thinking as a whole class. Although we recognize that mathematics teacher educators must often move quickly to cover a broad range of topics or are required to implement specific department-mandated curriculum materials, we contend that this lesson structure presents a potentially effective design feature for mathematical lessons across topics (Simon, 1994).

In related research, we have also observed that preservice teachers' attention to multiplicative structure supported the use of efficient methods for computing greatest common factors and least common multiples (Feldman, 2012). Future study is needed to investigate whether attention to multiplicative structure leads to deeper procedural knowledge of other K–12 topics, including operating with fractions, solving missing-value proportion problems, and factoring polynomials.

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Authors

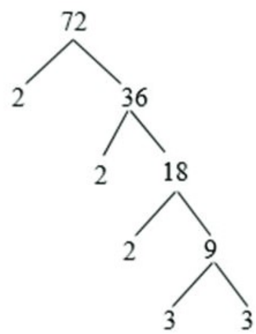
Ziv Feldman, Mathematics Education Program, School of Education 714, Boston University, Boston, MA 02215; zfeld@bu.edu

Matt B. Roscoe, Mathematical Sciences Department, Math 213, University of Montana, Missoula, MT 59812; matt.roscoe@umontana.edu

Appendix A: The Sequence of Lessons for the Curricular Intervention

Lesson 1

1. In a factor tree, numbers are repeatedly factored until one reaches a prime number at the end of each branch. Below is an example of a factor tree for 72. Create two different factor trees for 72.



(a) What is the same about all three factor trees? What is different about them?

(b) Use the factor trees above to write 72 as a product of prime numbers. This representation of 72 is called a prime factorization of 72.

2. Consider the following factorizations of *R*, *N*, and *K*:

$R = 3^2 \times 4^2 \times 5$

$K = 16 \times 45$

$N = 18 \times 40$

Without multiplying, determine which has the greatest value. Explain how you know.

Group Discussion Questions

- What is the difference between the prime factorization of a number and the factorization of that same number?
- The fundamental theorem of arithmetic states that any composite number can be decomposed into a product of prime numbers, and that this product is unique. What does this theorem mean?

Lesson 2

1. Briefly check your prehomework results with your group, making sure you agree on the prime factorizations in the grid. Also share any patterns and/or discoveries made during the prehomework. Write 2–3 discoveries that your group agrees on below.

2. (a) You are assigned the number ___________. Using your prehomework grid, use a highlighter to color the cells of all the factors of that number.

   (b) Once you have finished, make a list below of that number's factors and the prime factorization of each factor.

Factors of ____________	**Prime Factorizations**

   (c) How are the prime factorizations of the factors of your number related to the prime factorization of your number?

   (d) Does your conjecture from part (c) hold true for other numbers and their factors? Confer with other members of your group by examining their data from part (b).

Group Discussion Question

- How are the prime factorizations of the factors of a number, n, related to the prime factorization of n? Explain.

3. Consider the number $120 = 2^3 \times 3 \times 5$. Which of the following is a factor of 120? Briefly explain how you know for each answer.

   (a) 3 (b) 6 (c) 18

   (d) 30 (e) 35 (f) 40

4. I am thinking of a number less than 100; call it M. One of M's factors is 12.

   (a) What can you already say about M's prime factorization? Explain.

   (b) What numbers other than 12 are also factors of M? How do you know?

   (c) Suppose 5 is also a factor of M. What additional factors of M can you now identify?

5. Consider the number 68. Use the number's prime factorization to find all of its factors. Explain your process.

6. Let $K = 312 \times 832$ be the prime factorization of K. List every counting number that is a factor of K without computing the value of K. Explain briefly how to do this.

Group Discussion Questions

- Describe a method for finding the factors of a number using the number's prime factorization. Use the number 18 to illustrate this method.
- A traditional definition of a factor of a number N is as follows: "A is a factor of N if $N = A \times B$, where A and B are counting numbers." Use prime factorization to describe a new way to define a factor of a number.

Lesson 3

Use your grid from the prehomework to find 2 different numbers with 2, 3, 4, 5, and 6 factors. Compile your results with the rest of the class. How do you know they have the same number of factors?

Type of No.	Examples	Prime Factorizations
Numbers With 2 factors		
Numbers With 3 factors		
Numbers With 4 factors		
Numbers With 5 factors		
Numbers With 6 factors		

1. How many factors does each of the following numbers have? Explain without computing their values.

   (a) 2^2 (b) 2^3 (c) 2^4 (d) 2^n, where n is some counting number

2. (a) How many factors does $2^3 \times 3$ have?

   (b) How many factors does $2^3 \times 3^2$ have?

   (c) How many factors does $2^3 \times 3^3$ have?

   (d) Predict the number of factors for $2^3 \times 3^n$, where n is any positive integer. Explain your prediction.

3. Order the following numbers from having the greatest number of factors to having the least number of factors (explain any ties): 5^0, 5^1, 5^2, 5^3, 5^4, 5^5

Group Discussion Question

- What information does the prime factorization of a number provide about the number of factors it has? Use examples to illustrate your point.

Homework 1

Directions

Rewrite each integer below as a product of prime numbers (i.e., prime factorization). You are encouraged to use any method you want to complete this task. Be prepared to explain the process that you used to complete this task. Then answer the follow-up questions.

91	92	93	94	95	96	97	98	99	100
81	82	83	84	85	86	87	88	89	90
71	72	73	74	75	76	77	78	79	80
61	62	63	64	65	66	67	68	69	70
51	52	53	54	55	56	57	58	59	60
41	42	43	44	45	46	47	48	49	50
31	32	33	34	35	36	37	38	39	40
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	12

1. Explain how you completed the grid. Use examples to illustrate your method(s).

2. Describe any discoveries you made or patterns you noticed when completing the grid.

Homework 2

Directions

Answer all questions below in the spaces provided. Show all calculations and explain your responses thoroughly. **Answers without adequate explanations or work shown will not receive credit.**

1. Consider a number, N, whose prime factorization is $N = 2^5 \times 3^2 \times 7 \times 11$. Without computing the value of N, decide if each of the following numbers is a factor of N. Explain each choice briefly.

   (a) 3 (c) 18 (e) 35

   (b) 6 (d) 30 (f) 40

2. If a fifth grader states, "Larger numbers have more factors than smaller numbers," what numbers will you give him to investigate? Why those numbers?

3. A number, X, has 10 and 3 as two of its factors. What other number(s) must be factors of X? How do you know?

4. Is $3^2 \times 2^4$ a factor of $3^3 \times 2^2$? Explain why or why not.

5. Consider the number $M = 31 \times 7^2 \times 23$, where 31, 7, and 23 are prime numbers. List all the factors of M without computing the value of M first. Clearly show or explain how you were able to find all of the factors.

6. (a) If both 3 and 5 are factors of a number x, must 15 be a factor of x? Why or why not?

   (b) If 4 and 6 are factors of a number y, must 24 be a factor of y? Why or why not?

7. Jennifer and Billie are having an argument. Both have constructed a factor tree for 132. Jennifer claims that the two trees are the same. Billie claims that the two trees are different. What, do you suppose, is the source of their confusion? Give examples. How would you resolve their argument?

Appendix B: Pretest and Posttest Scoring Rubrics

Pretest Scoring Rubric

Total Points = 25

1. Consider the number $N = 3^2 \times 5^4 \times 11 \times 17^3$. Without calculating the value of N, determine whether each of the following is a factor of N. Justify each decision briefly.

(a) 5 (c) 15 (e) 75

(b) 19 (d) 21

Suggested Solutions

(a) Yes, because 5 is in the prime factorization of N (argument 1).

(b) Yes, because $5 \times (3^2 \times 5^3 \times 11 \times 17^3) = N$, so $N \div 5$ results in a whole number (argument 2).

(c) No, because 19 is not in the prime factorization of N.

Yes, because 15 is in the prime factorization of N as 3×5 (argument 1); yes, because $N = (3 \times 5) \times (3 \times 5^3 \times 11 \times 17^3) = 15 \times (3 \times 5^3 \times 11 \times 17^3)$, so $N \div 15$ results in a whole number (argument 2).

(d) No, because $21 = 3 \times 7$ but there is no 7 in the prime factorization of N, so 21 is not in the prime factorization of N.

(e) Yes, because 75 is in the prime factorization of N as $3 \times 5 \times 5$ (argument 1); yes, because $N = (3 \times 5 \times 5) \times (3 \times 5^2 \times 11 \times 17^3) = 75 \times (3 \times 5^2 \times 11 \times 17^3)$, so $N \div 75$ results in a whole number (argument 2).

Scoring Criteria (10 possible points):

- 1 point for **each** correct (yes/no) response
- 1 point for **each** correct and complete argument (only one argument is needed). The argument should make reference to the prime factorization to be complete (e.g., it is not sufficient to say "19 is not a factor because 19 is not a factor" or "5 is a factor because 5^2 is in the number").

2. (a) List all the factors of 225. Show how you found all of them.

Suggested Solution

(a) 1, 3, 5, 9, 15, 25, 45, 75, and 225. Demonstration of reasoning can include at least one of the following: long division, factor trees, prime factorization combinations, factor pairs, or any reasonable method.

Scoring Criteria (5 possible points):

- 3 points possible for identifying factors
  - Give 3 points if 9 factors are correctly identified.
  - Give 2 points if 7–8 factors are correctly identified.
  - Give 1 point if 3–6 factors are correctly identified.
  - Give 0 points if 0–2 factors are correctly identified.
  - Note: Net out any incorrect factors from the number of correct factors to determine how many points should be recorded.

- 2 points possible for demonstration of reasoning
  - Give 2 points if work is correct and clearly and explicitly shows how the list of factors was determined.
  - Give 1 point if work shown is a reasonable method but there is limited clarity around how the list of factors was determined.
  - Give 0 points if work is not shown or is incorrect or no list of factors is generated.

(b) List all the factors of $5^2 \times 7^2$. Show how you found all of them.

Suggested Solution

(b) 1, 5, 7, 5^2, 7^2, 5×7, $5^2 \times 7$, 5×7^2, and $5^2 \times 7^2$. Demonstration of reasoning must employ the use of prime factorization to find prime factor combinations.

Scoring Criteria (5 possible points):

- 3 points possible for identifying factors
  - Give 3 points if all 9 factors are correctly identified.
  - Give 2 points if 7–8 factors are correctly identified.
  - Give 1 point if 3–6 factors are correctly identified.
  - Give 0 points if 0–2 factors are correctly identified.
  - Note: Net out any incorrect factors from the number of correct factors to determine how many points should be recorded.

- 2 points possible for demonstration of reasoning
  - Give 2 points if work correctly uses uncoordinated and coordinated prime factor combinations to clearly and explicitly show how the list of factors was determined.
  - Give 1 point if work shown uses prime factor combinations but there is limited clarity about how the list of factors was determined or either uncoordinated or coordinated prime factor combinations are missing.
  - Give 0 points if prime factor combinations are not used, work is incorrect, or no list of factors is generated.

3. What is the smallest positive integer that has the first ten counting numbers, 1–10, as its factors? Show or explain your work so that others can follow your logic. Note: You may leave your answer in factored form.

Suggested Solution

The smallest positive integer that is divisible by all of the first ten counting numbers is 2,520. Since the number must be divisible by 1–10, we can build its prime factorization to include all these numbers as factors. The prime factorization is as follows: $2^3 \times 3^2 \times 5 \times 7$. In this prime factorization, we see that 2, 3, 5, and 7 are factors because they are prime factors of the number. Since 2^3 is part of the prime factorization of the number, 4 and 8 are also factors. Since 3^2 is represented, 9 is also a factor. Since 2 and 3 are represented as prime factors, then 6 is also a factor of the number, and since 2 and 5 are represented as prime factors of the number, 10 is also one of its factors. As always, 1 is a factor of any whole number. The number 2,520 is the smallest positive integer that satisfies the given criterion because if any of the prime factors are removed from its prime factorization, at least one of the first ten counting numbers will cease to be its factor.

Scoring Criteria (5 possible points):

- 3 points for correctly identifying the solution as 2,520 OR as $2^3 \times 3^2 \times 5 \times 7$.
    - Give 2 points for this section if solution has 1 error (e.g., gives close to correct prime factorization but includes 2^4 or 3^3 or leaves out a 7).
    - Give 1 point for this section if solution has 2–3 errors (e.g., includes a 2^4 and a 3^3 in the prime factorization, or just multiplies $1 \times 2 \times 3 \times . . . \times 9 \times 10 = 3{,}628{,}800$.)
    - Give 0 points for this section if solution has more than 3 errors.

- 2 points for correct reasoning.
    - Give 2 points if reasoning is correct and complete (i.e., the work shown and/or explanation given fully explain why the provided solution is correct).
    - Give 1 point if reasoning is partially correct or incomplete (i.e., the work shown and/or explanation given do not address all necessary points or shows incorrect reasoning at times).
    - Give 0 points if reasoning is nonexistent or completely incorrect (i.e., the work shown and/or explanation given do not address any necessary points or shows incorrect reasoning throughout).

Posttest Scoring Rubric

Total Points = 25

1. Consider the number $N = 2^3 \times 5^4 \times 7^2 \times 13$. Without calculating the value of N, determine whether each of the following is a factor of N. Justify each decision briefly.

   (a) 11

   (b) 7

   (c) 14

   (d) 21

   (e) 98

Suggested Solutions

(a) No, because 11 is not in the prime factorization of N.

(b) Yes, because 7 is in the prime factorization of N (argument 1); yes, because $7 \times (2^3 \times 5^4 \times 7 \times 13) = N$, so $N \div 7$ results in a whole number (argument 2).

(c) Yes, because 14 is in the prime factorization of N as 2×7 (argument 1); yes, because $N = (2 \times 7) \times (2^2 \times 5^4 \times 7 \times 13) = 14 \times (2^3 \times 5^4 \times 7 \times 13)$, so $N \div 14$ results in a whole number (argument 2).

(d) No, because $21 = 3 \times 7$, but there is no 3 in the prime factorization of N, so 21 is not in the prime factorization of N.

(e) Yes, because 98 is in the prime factorization of N as $2 \times 7 \times 7$ (argument 1); yes, because $N = (2 \times 7 \times 7) \times (2^2 \times 5^4 \times 13) = 98 \times (2^2 \times 5^4 \times 13)$, so $N \div 98$ results in a whole number (argument 2).

Scoring Criteria (10 possible points):

- 1 point for each correct (yes/no) response
- 1 point for each correct and complete argument (only one argument is needed). The argument should make reference to the prime factorization to be complete (e.g., it is not sufficient to say “11 is not a factor because 11 isn’t a factor” or “7 is a factor because 7^2 is in the number”).

2. (a) List all the factors of 300. Show how you found all of them.

Suggested Solution

(a) 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 25, 30, 50, 60, 75, 100, 150, and 300. Work shown can include at least one of the following: long division, factor trees, prime factorization combinations, factor pairs, or any reasonable method.

Scoring Criteria (5 possible points):

- 3 points possible for identifying factors
  - Give 3 points if all 18 factors are correctly identified.
  - Give 2 points if 14–17 factors are correctly identified.
  - Give 1 point if 3–13 factors are correctly identified.

- o Give 0 points if 0–2 factors are correctly identified.
- o Note: Net out any incorrect factors from the number of correct factors to determine how many points should be recorded.

- 2 points possible for demonstration of reasoning
  - o Give 2 points if work is correct and clearly and explicitly shows how the list of factors was determined.
  - o Give 1 point if work shown is a reasonable method but there is limited clarity about how the list of factors was determined.
  - o Give 0 points if work is not shown or is incorrect or no list of factors is generated.

2. (b) List all the factors of $5^2 \times 7 \times 13^2$. Show how you found all of them.

Suggested Solution

(b) 1, 5, 7, 13, 5^2, 13^2, 5×7, 5×13, 7×13, $5^2 \times 7$, $5^2 \times 13$, 5×13^2, 7×13^2, $5 \times 7 \times 13$, $5^2 \times 7 \times 13$, $5 \times 7 \times 13^2$, $5^2 \times 13^2$, and $5^2 \times 7 \times 13^2$. Work shown must center around using the prime factorization to find prime factor combinations.

Scoring Criteria (5 possible points):

- 3 points possible for identifying factors
  - o Give 3 points if all 18 factors are correctly identified.
  - o Give 2 points if 14–17 factors are correctly identified.
  - o Give 1 point if 3–13 factors are correctly identified.
  - o Give 0 points if 0–2 factors are correctly identified.
  - o Note: Net out any incorrect factors from the number of correct factors to determine how many points should be recorded.

- 2 points possible for demonstration of reasoning
  - o Give 2 points if work correctly uses uncoordinated and coordinated prime factor combinations to clearly and explicitly show how the list of factors was determined.
  - o Give 1 point if work shown uses prime factor combinations but there is limited clarity around how the list of factors was determined or either uncoordinated or coordinated prime factor combinations are missing.
  - o Give 0 points if prime factor combinations are not used, work is incorrect, or no list of factors is generated.

3. What is the smallest positive integer that has the first ten counting numbers, 1–10, as its factors? Show or explain your work so that others can follow your logic.

   Note: you may leave your answer in factored form.

Suggested Solution

The smallest positive integer that is divisible by all of the first ten counting numbers is 2,520. Since the number must be divisible by 1–10, we can build its prime factorization to include all these numbers as factors. The prime factorization is as follows: $2^3 \times 3^2 \times 5 \times 7$. In this prime factorization, we see that 2, 3, 5, and 7 are factors because they are prime factors of the number. Since 2^3 is part of the prime factorization of the number, 4 and 8 are also factors. Since 3^2 is represented, 9 is also a factor. Since 2 and 3 are represented as prime factors, then 6 is also a factor of the number, and since 2 and 5 are represented as prime factors of the number, 10 is also one of its factors. As always, 1 is a factor of any whole number. The smallest positive integer that satisfies the given criterion is 2,520 because if any of the prime factors are removed from its prime factorization, at least one of the first ten counting numbers will cease to be its factor.

Scoring Criteria (5 possible points):

- 3 points for correctly identifying the solution as 2,520 **or** as $2^3 \times 3^2 \times 5 \times 7$.
  - Give 2 points for this section if solution has 1 error (e.g., gives close-to-correct prime factorization but includes 2^4 or 3^3 or leaves out a 7).
  - Give 1 point for this section if solution has 2–3 errors (e.g., includes a 2^4 and a 3^3 in the prime factorization, or just multiplies $1 \times 2 \times 3 \times \ldots \times 9 \times 10 = 3{,}628{,}800$).
  - Give 0 points for this section if solution has more than 3 errors.

- 2 points for correct reasoning.
  - Give 2 points if reasoning is correct and complete (i.e., work shown and/or explanation given fully explain why the provided solution is correct).
  - Give 1 point if reasoning is partially correct or incomplete (i.e., work shown and/or explanation given do not address all necessary points or shows incorrect reasoning at times).
  - Give 0 points if reasoning is nonexistent or completely incorrect (i.e., work shown and/or explanation given do not address any necessary points or shows incorrect reasoning throughout).