Lesson Plan Overview

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<th>Timing</th>
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| Problem 1   | Problem 1 has participants draw 5 different parallelograms each with a base of 4 and a height of 3. It then asks them to explain why the area formula for a parallelogram is $A=bh$. Key ideas built upon in this problem include:  
  - Understanding the area formula for a rectangle helps us derive the area formula for a parallelogram.  
  - Shapes can be decomposed and recomposed without changing overall area.  
|              | The DQ is another opportunity for participants to articulate their reasoning for the area formula of a parallelogram.                                                                                                                                                                                                                      | **Launch:** Tell participants that they will explain and make sense of the area formulas for parallelograms and triangles in this lesson by using some of the general strategies they developed in the *Area Concepts* lesson. Also mention that once an area formula is derived (e.g., for a rectangle), it can be used to derive the area formula for another figure (e.g., for a parallelogram).  

**P1:** Direct groups to work on Problem 1. If you notice groups are struggling to draw different parallelograms, ask them to consider the main characteristics of a parallelogram (i.e., parallel and congruent opposite sides). For Problem 1b, remind them to review the strategies they explored in the *Area Concepts* lesson. Take note of their strategies, as several different approaches typically come up (See Video 1). Select 1-3 to have them present during whole class discussion.  

**DQ:** Facilitate a whole class discussion in which you ask different groups to present their justifications. Push them to generalize their reasoning to parallelograms with base $b$ and height $h$ (See Videos 2-3). |
| Whole Class Discussion Question (10 min) |                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                   |
| Problem 2   | Problem 2 has participants use triangular cutouts to explain the area formula for a triangle, $A = \frac{1}{2}bh$, using what they know about the areas of parallelograms and rectangles. Participants use a variety of strategies including combining two congruent triangles to form a parallelogram to explain the formula.  
|              | The DQ has participants explain why the formula for the area of a triangle is correct for all triangles.                                                                                                                                                                                                                               | **P2:** Instruct participants to use the triangular cutouts to explore ways of explaining the area formula for a triangle. Some will decompose the triangles and recompose them. Others will combine 2 congruent triangles to make a parallelogram. Sometimes it helps to list the strategies studied so far on the board. If participants are stuck, we have joined a table and combined 2 of the congruent triangles into a parallelogram and then asked: “Will this work for all triangles? Could this be used to explain the formula?”  

**DQ:** Participants provide a convincing argument that the formula for the area of any triangle is $A = \frac{1}{2}bh$. Participants should be pressed to provide precise mathematical justifications (See Video 5). |
| Whole Class Discussion Question (10 min) |                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                   |
Teaching the Lesson

Learning Goals
This lesson has participants derive area formulas for parallelograms and triangles and explain why those formulas make sense. Each derivation uses the some of the general strategies for finding area explored in the Area Concepts lesson. Additionally, the derivations build on previous derivations (e.g., the area formula for a parallelogram builds on the area formula for a rectangle) so that participants can make connections among the area formulas.

Materials
- *Parallelograms and Triangles* lesson
- *Parallelograms and Triangles Homework*
- Grid paper (attached to lesson)
- 8 pre-cut triangles for each pair or small group of participants.
- Scissors, tape, colored markers or crayons, and rulers.
- Document camera, if available.

Timing
<table>
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<th>Time</th>
<th>Activity</th>
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<tbody>
<tr>
<td>50 Minutes</td>
<td>All problems and summary.</td>
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<tr>
<td>80 Minutes</td>
<td>Add discussion of additional strategies for deriving the area formulas of parallelograms and triangles.</td>
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</tbody>
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Launching the Lesson
Introduce the lesson by telling participants that they will be exploring two common area formulas that they most likely learned in the past. Explain that even though they may know these formulas, the goal of the lesson is to understand why these formulas are true and how they relate to one another. Consider also reminding them that such understanding is a critical aspect of effective teaching, as teachers need to possess deep and flexible knowledge of mathematics in order to teach students effectively.

Before you begin, you may need to review how to use grid paper. Specifically, participants typically have two misconceptions. First, when measuring length, participants may be attending to the squares above a line instead of the individual line segments. For example, in the following image, they may claim that the length of the base is three and a half.
Tell participants that when we measure length, we iterate linear units. Draw a parallelogram on grid paper and run your finger along each of the line segments instead of the squares, as you count to determine the length of the base. You may want to show them an example similar to the one above as an example of an elementary student’s misconceptions and ask them what the student is thinking and why it is incorrect to say that the base measures 3 ½ units.

Second, participants believe they can measure line segments that cut through squares diagonally by counting the number of squares the segment passes through. In the example above, such a participant might claim that the left side of the parallelogram measures 2 units, even though it is actually longer. You can have participants use string, measuring tapes, or rulers to convince themselves that the height of the parallelogram is shorter than the side length. Indeed, participants may need to be explicitly told that the height is perpendicular to the base and see some examples of incorrectly identified heights.

Direct groups to work on Problem 1; explain that the general strategies they learned in the Area Concepts lesson may be useful to them. Also mention that they can use the area formula for a rectangle to help them, since they have already explained that formula in the Area and Perimeter lesson.

**Problem 1**

1. a) On the attached grid paper draw 5 different parallelograms with a base of 4 units and a height of 3 units. One of these parallelograms should be a rectangle. Determine their areas.

   b) Apply some of the strategies you learned in previous lessons to explain how you know for sure the areas of the parallelograms in Problem 1a. Consider why the area of these parallelograms can be determined using the formula, $A=bh$, where $b$ is the length of the base and $h$ is the height.

In previous enactments, we have found that many participants are very unfamiliar with how to draw parallelograms. Many are able to draw a rectangle and one non-rectangular parallelogram with a base of 4 units and a height of 3 units, but they struggle to recognize that additional parallelograms are possible. In these cases, ask participants to consider the characteristics of parallelograms; or suggest they start by drawing a length of 4 units for one base and then decide where the other parallel base can be located (and how the height of 3 units is involved).
To derive the area formula, groups might consider two cases of parallelograms: 1) those where the height is located inside the parallelogram, and 2) those where the height is located outside the parallelogram. Let’s examine the first case. The technique of dissection, also referred to as decomposition or subdivision, of the parallelogram into pieces can be used to explain why the area is equal to base × height. Subdivide a parallelogram into a trapezoid and a right triangle, as shown below on the left. Since a parallelogram’s opposite sides are congruent and parallel, one can translate the right triangle to the opposite side of the parallelogram and recompose it, as shown in the second figure on the right. The result is a rectangle with the same length base and height as the original parallelogram. Thus, the parallelogram’s area will equal the rectangle’s area.

In the other case, a parallelogram’s height is located outside of the shape, not inside as above. Sometimes simply rotating a parallelogram so a different side is designated as the base changes where the height is located. Or, one can enclose the parallelogram in a rectangle. In the drawing below, this rectangle is composed of two copies of a right triangle with bases “a” and the parallelogram (shaded).
The rectangle has area \((a + b) \times h\) which is equal to \(a \times h + b \times h\) (using the distributive property). If you compose the triangles together, they form a rectangle with an area of \(a \times h\). Now subtract the area of this rectangle composed of triangles from the area of the larger rectangle:

\[
(a \times h + b \times h) - (a \times h) = b \times h
\]

Thus, the area of the parallelogram is shown to be \(b \times h\).

In our experience, participants often use the dissection technique described above. See Video 1 (http://elementarymathproject.com/emp_lesson/parallelograms-and-triangles/) for an example of a small group trying to use this technique. Some also decompose a parallelogram into two right triangles and a rectangle and use algebraic notation to arrive at an area of \(b \times h\). In this case, participants may use the area formula for a triangle to derive the area formula for a parallelogram. If you see this happening, ask participants how they know the area formula for a triangle is \(\frac{1}{2}bh\). If they say that a triangle is half the area of a parallelogram, remind them that they have not yet derived the area formula for a parallelogram. However, if they say that a right triangle is half the area of a rectangle, then this is appropriate reasoning since they have already explained the area formula for a rectangle in the Area and Perimeter lesson. In preparation for the whole class discussion and depending on how much time you have left in the period, select 1-3 group-generated strategies to have presented. Make sure that the dissection technique is one of those strategies.

**Whole Class Discussion Question**

- Provide a convincing argument for why the area of any parallelogram is the product of base length and height.

Facilitate a whole class discussion in which participants justify the area of a parallelogram using the dissection technique described above and any additional strategies (if time allows). Video 2 (http://elementarymathproject.com/emp_lesson/parallelograms-and-triangles/) shows a whole class discussion in which participants present and compare two different methods for Justifying the area formula of a parallelogram. As different groups present their strategies, tell the rest of the class that they should be asking the presenting group questions if they have any. Oftentimes, participants will not ask questions even if they do not understand the presented justification. In this case, you should ask the presenters some clarifying questions that you want them to answer and others to hear. For example, explaining why any parallelogram can be decomposed and recomposed into a rectangle with a width equal to the parallelogram’s height and a length equal to the base of the parallelogram is an essential part of the justification that many participants do not even recognize is important to explain. The justification for this is that the opposite sides of parallelograms are parallel and congruent; the two parts of a parallelogram will fit flush, with no gaps or overlap. Ask presenters, “How do you know that when you move the right triangle over to the other side of the trapezoid you get a rectangle? How do you know that this right triangle fits perfectly on the other side?” If participants struggle to explain this, ask them to consider how the properties of a parallelogram play a role in creating the rectangle. See Video 3 (http://elementarymathproject.com/emp_lesson/parallelograms-and-triangles/) for an example of how to facilitate such a conversation. Once the class has developed at least one derivation and explained it thoroughly, continue to Problem 2.
**Problem 2**

Direct groups to work on Problem 2. Tell them that they should use the triangular cutouts to develop a strategy. Remind them that now that they have derived the area formula for a parallelogram, they can use it to help derive the area formula for a triangle.

2. Cut out 2 copies of each of the following triangles from the end of this packet:
   - right triangle
   - isosceles triangle
   - scalene triangle
   - equilateral triangle

   Your goal is to explain to others why the formula for the area of a triangle \( A = \frac{1}{2}bh \) makes sense. Your reasoning must work for any type of triangle. Consider using some of the strategies for determining area that you learned in prior lessons.

In our experience, one derivation that always arises is where two congruent triangles are manipulated to form a parallelogram by rotating one triangle 180° and composing it with the second triangle. Since corresponding sides of congruent triangles are congruent, rotating one triangle by 180° results in one pair of corresponding sides lining up with each other. Since corresponding angles of congruent triangles are congruent, this rotation and subsequent composition results in a quadrilateral with opposite angles congruent – a parallelogram.

![Diagram of two triangles forming a parallelogram](image)

The formula for the area of a triangle is \( A = \frac{1}{2}bh \). This formula makes sense because two congruent triangles can be manipulated to form a parallelogram whose area is base \( x \) height, as shown above. As such, the area of each triangle is one-half the area of the corresponding parallelogram. Since the base of the parallelogram is the base of each triangle and the height of the parallelogram is the height of each triangle, the area of one triangle can be determined using the formula, \( A = \frac{1}{2}(\text{base} \times \text{height}) \).

Again, as you circulate from group to group, listen for justifications of why the recomposed triangles form a parallelogram. For those who struggle, ask them to consider the properties of a parallelogram and if the newly composed figure possesses some of those properties. As you consider which groups to select for presentation, identify groups that do and do not include this part of the justification. Ask a group that does not include this justification to present first. See Video 4 ([http://elementarymathproject.com/emp_lesson/parallelograms-and-triangles/](http://elementarymathproject.com/emp_lesson/parallelograms-and-triangles/)) for a small
group working together to construct this justification. Other derivations may also be suggested; select groups to share their approaches and justifications, time permitting.

**Whole Class Discussion Question**

- Explain why your formula for the area of a triangle is correct for all triangles.

We have found that verbal descriptions of the rotation of the triangular cutouts can be confusing without a visual depiction. If available, have participants present this work under a document camera. It can also help to have the angles and sides of the triangular cutouts labeled so that when discussing why the newly recomposed figure is a parallelogram, specific angles and sides can easily be identified and discussed.

During whole-class discussion, participants may have difficulty making clear and precise mathematical justifications of their area formulas. One strategy for helping them clarify their meaning to themselves and to others is by “revoicing.” An instructor can revoice a participant’s comment by restating it in a slightly different way and then asking the participant if the restatement conveys the original message accurately. Another strategy involves asking others in the class to restate the comment in their own words. We have found that it may take several iterations of participant contribution, revoicing, and restating in order for the class as a whole to make sense of these formulas. Video 5 (http://elementarymathproject.com/emp_lesson/parallelograms-and-triangles/) shows a good example of participants engaging in mathematical justifications around the area formula of a triangle.

**Parallelograms and Triangles Homework: Summarize & Connect**

Assign the following two problems to participants for homework to solidify their understanding of the key ideas of the lesson. If you wish to assign additional homework questions, they can be found in the *Parallelograms and Triangles Homework* document.

1. Explain why the area formula for a parallelogram makes sense.

2. Explain why the area formula for a triangle makes sense.